

Imperial/TP/95-96/3
 hep-ph/9510298
 15th October, 1995

THE CONDENSED MATTER LIMIT OF RELATIVISTIC QFT *

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ABSTRACT

I study how to apply relativistic quantum field theory to condensed matter systems. The motivation for this is examined and then two separate elements are considered. First we identify the precise relationship between relativistic and non-relativistic fields. Second we consider the need for a chemical potential and how one includes this in static and dynamical calculations.

1. Introduction

There has been considerable recent interest in condensed matter systems amongst cosmologists^{1,2,3}. This is because it seems possible to apply methods used in the early universe to some condensed matter systems. The latter calculations can be compared to laboratory experiments which in turn can give us further confidence in the early universe results. The table below illustrates some of the links which I will discuss in the paper. In particular we note the

	Cosmology	Condensed Matter
Theory used	Relativistic QFT	Non-Relativistic QFT
Parameter Ranges	$ \mu \ll T \sim m$	$T \sim m - \mu = -\mu_{\text{nr}} \ll \mu \sim m$
Example Theory	GUTS	Liquid $^3\text{He}, ^4\text{He}$
Example defects	U(1) Cosmic Strings	Vortices
Equation of Motion	$(\partial_t^2 + k^2 + m^2)\Phi = 0$	$(i\partial_t - \frac{k^2}{2m^2} - \mu_{\text{nr}})\Psi = 0$
Fundamental Field	$\Phi = \frac{1}{\sqrt{2\omega}}[\Psi + \bar{\Psi}^*]$ $\Pi = i\sqrt{(\frac{\omega}{2})}[\Psi^* - \bar{\Psi}]$	$\Psi = \sqrt{\frac{\omega}{2}}\Phi + \frac{i}{\sqrt{2\omega}}\Pi^*$ $\bar{\Psi} = \sqrt{\frac{\omega}{2}}\Phi^* + \frac{i}{\sqrt{2\omega}}\Pi$

Table 1. Analogies between Relativistic and Condensed Matter systems.

theoretical work on cosmic strings, and the links made¹ between this and the experiments⁵ on vortices on liquid helium, both ^3He and ^4He . For example there is the recent work using out-of-equilibrium quantum field theory on the creation of cosmic strings⁶. In this a single complex scalar field was used and it was important to identify the correct physical dynamical behaviors, especially the mass, to be used in the calculations. It would be interesting to look

*Talk given at the 4th Workshop on Thermal Field Theories and their Applications, 6th-10th August 1995, Dalian, China. To appear in the proceedings, to be published by World Scientific. This talk available via anonymous FTP from euclid.tp.ph.ic.ac.uk as papers/95-6.3.tex and L^AT_EXsource and postscript versions available via WWW from <http://euclid.tp.ph.ic.ac.uk/Papers/index.html>.

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at the condensed matter limit as this can be considered as a toy model for liquid ^4He . However we need to identify precisely what are the non-relativistic fields in terms of the relativistic fields.

First we must specify what we meant by the relativistic and condensed matter limits. The energy scales involved in the relativistic problems are such that a particle with typical thermal kinetic energy is moving at relativistic speeds, so $T \gtrsim m$. Then the Hamiltonian used is built around free particles with the usual dispersion relation $\omega(k) = \sqrt{k^2 + m^2}$. In more practical terms, this usually means that thermal fluctuations are of order the energy scale of the theory, so this is where there is likely to be interesting physics e.g. phase transitions. In many cosmological problems, the density of relativistic particles, n_+ , and relativistic anti-particles, n_- , is large but essentially the same. In equilibrium we have

$$n_{\pm}(k) = \frac{1}{\exp\{\beta(\omega(k) \mp \mu)\}}, \quad (1)$$

so this means that we have a tiny chemical potential,

$$\left| \frac{n_+ - n_-}{n_+ + n_-} \right| \ll 1 \quad \Rightarrow \quad \frac{|\mu|}{m} \ll 1 \quad (2)$$

For instance at high temperatures in the early universe the ratio of the baryon/anti-baryon asymmetry to the baryon density is around 10^{-10} .

The condensed matter limit is a non-relativistic limit so that $T \ll m$ and for free particles the dispersion relation is $\omega(k) \simeq m + k^2/(2m)$. For instance in liquid Helium we are working at $T \sim 1\text{K}$ or less. However, the relevant particles are the Helium atoms whose masses are equivalent to 10^{13}K . The other aspect of the condensed matter limit, and a further difference from typical relativistic problems, is that there are essentially no anti-particles so that

$$\frac{n_-(k)}{n_+(k)} \ll 1 \quad \Rightarrow \quad \mu \simeq m. \quad (3)$$

It is for this reason that condensed matter texts use a different definition for chemical potentials. Here I will denote this alternative by μ_{nr} . The relationship to the chemical potential, μ , normally used in relativistic texts, is given by

$$\mu_{\text{nr}} := \mu - m \gg -m. \quad (4)$$

Note that for Fermi systems, μ_{nr} has a direct physical meaning in terms of the Fermi surface as it is the Fermi energy at zero temperature and plays a similar role at non-zero temperatures. For Bosons with no condensation present, $|\mu|$ is strictly less than the physical mass m and it has no direct physical interpretation in terms of some ‘Bose-surface’. With Bose-Einstein condensation, $|\mu| = m$ exactly and μ is no longer even a free parameter. The treatment of this case is more intricate. I will focus on Bosonic systems in this talk.

2. Extracting the non-relativistic fields

In looking at relativistic and non-relativistic bosonic systems, we note that the usual equations of motion for the free fields differ by being second order and first order in time derivatives respectively;

$$(\partial_t^2 + k^2 + m^2)\Phi = 0, \quad (5)$$

$$(-i\partial_t + \frac{k^2}{2m^2} - \mu_{\text{nr}})\Psi = 0. \quad (6)$$

Note (6) is the Gross-Pitaevskii equation rather than the time-dependent Ginzburg-Landau equation¹. The latter has no factor of i in front of the time derivative but is generally thought by condensed matter theorists to be a better model for systems like ^4He . This suggests that in trying to express the condensed matter fields Ψ in terms of the relativistic fields we must split up the relativistic Klein-Gordon equation into two first order (in time) equations. This is sometimes done when solving differential equations numerically, when one would take Φ and $\partial_t\Phi$ as independent variables. So we will work with Φ and its conjugate variable Π ($\sim \partial_t\Phi$) and start not from the usual Lagrangian path integral but from the Hamiltonian version. In this way, Π remains available as an independent variable which we can use, whereas it is integrated out in reaching the Lagrangian formulation. Thus a suitable generating functional to consider is

$$Z[j, j^*; \mu] = \int_B \mathcal{D}\Pi^* \mathcal{D}\Pi \mathcal{D}\Phi^* \mathcal{D}\Phi \exp \left(i\tilde{S}_e[\Pi, \Phi; \mu] + i \int (j^*\Phi + j\Phi^*) \right), \quad (7)$$

where

$$B \equiv \{\Phi(t) = \Phi(t - i\beta)\} \quad (8)$$

$$\tilde{S}_e[\Pi, \Phi; \mu] = \int_0^{-i\beta} dt \left[\int d^3x (\Pi \partial_t \Phi + \Pi^* \partial_t \Phi^*) - (H - \mu Q) \right], \quad (9)$$

$$H = \int d^3x [\Pi^* \Pi + (\nabla \Phi^*)(\nabla \Phi) + m^2 \Phi^* \Phi + O(\lambda)] \quad (10)$$

$$Q = i \int d^3x (\Phi^* \Pi^* - \Pi \Phi). \quad (11)$$

and $O(\lambda)$ represents all cubic and higher order terms. It is useful to think of S_e as a matrix

$$\begin{aligned} & \tilde{S}_e[\Pi, \Phi; \mu] \\ &= \int_0^{-i\beta} dt \int d^3x \left(\sqrt{\omega} \Phi^*, \frac{-i}{\sqrt{\omega}} \Pi \right) \begin{pmatrix} -\omega & i\partial_t + \mu \\ i\partial_t + \mu & -\omega \end{pmatrix} \begin{pmatrix} \sqrt{\omega} \Phi \\ \frac{i}{\sqrt{\omega}} \Pi^* \end{pmatrix}. \end{aligned} \quad (12)$$

Clearly proceeding straight from here produces the usual relativistic formalism. So what we have to do is to replace the four fields, Φ, Π and their conjugates, by four new ones. We do this by finding a canonical transformation which leaves us with a temporal first order action and which is diagonal in the new fields up to and including the quadratic terms. This is sufficient to specify the new combinations to be

$$\Psi = \sqrt{\frac{\omega}{2}} \Phi + \frac{i}{\sqrt{2\omega}} \Pi^* \quad (13)$$

$$\bar{\Psi} = \sqrt{\frac{\omega}{2}} \Phi^* + \frac{i}{\sqrt{2\omega}} \Pi \quad (14)$$

together with their complex conjugates. This linear transformation also means that the measure in the path integral does not pick up any strange terms

$$\mathcal{D}\Pi^*\mathcal{D}\Pi\mathcal{D}\Phi^*\mathcal{D}\Phi = \mathcal{D}\Psi^*\mathcal{D}\Psi\mathcal{D}\bar{\Psi}^*\mathcal{D}\bar{\Psi}. \quad (15)$$

Physically what has been achieved is easy to see in terms of free fields as we have that

$$\Psi_{\text{free}} \sim \int d^3\vec{k} a(\vec{k})e^{-i\vec{k}\cdot\vec{x}+i\omega t}, \quad \bar{\Psi}_{\text{free}} \sim \int d^3\vec{k} b(\vec{k})e^{-i\vec{k}\cdot\vec{x}+i\omega t} \quad (16)$$

where $a(\vec{k}), b(\vec{k})$ annihilate particles and anti-particles respectively.

Putting this all together we find that the generating functional now looks like

$$Z[j, j^*; \mu] = \int \mathcal{D}\Psi^*\mathcal{D}\Psi\mathcal{D}\bar{\Psi}^*\mathcal{D}\bar{\Psi} \exp\{\tilde{S}_{\mu, \text{nr}, e}\} \quad (17)$$

$$\begin{aligned} \tilde{S}_{\mu, \text{nr}, e} &= S_{0, \text{nr}}[\Psi^*, \Psi; \mu, \omega] + S_{0, \text{nr}}[\bar{\Psi}^*, \bar{\Psi}; -\mu, \omega] \\ &\quad + S_{I, \text{nr}}[\Psi^*, \Psi, \bar{\Psi}^*, \bar{\Psi}] \end{aligned} \quad (18)$$

$$\tilde{S}_{0, \text{nr}}[\Psi^*, \Psi, \mu, \omega] = \int_0^{-i\beta} dt \int d^3x \Psi^* \left[i \frac{\partial}{\partial t} - \omega(\nabla) + \mu \right] \Psi \quad (19)$$

on removing total derivatives and in the last expression doing a small momentum expansion. Thus the quadratic part of the non-relativistic fields Ψ and $\bar{\Psi}$ has a global $O(2) \otimes O(2)$ symmetry. Particles and anti-particles are separately conserved under this symmetry. However typical interaction terms, contained in $S_{I, \text{nr}}$, break this symmetry, e.g.

$$\lambda |\Phi^* \Phi|^2 = \frac{\lambda}{4\omega^2} (\Psi^* \Psi + \bar{\Psi}^* \bar{\Psi} + \bar{\Psi}^* \Psi + \Psi^* \bar{\Psi}). \quad (20)$$

Note that the global $O(2) \otimes O(2)$ symmetry of non-interacting $\Psi, \bar{\Psi}$ fields is a mixture of the original Poincaré symmetries and the original global $O(2)$ of the Φ fields. Put another way, the $O(2) \otimes O(2)$ symmetry transformations written in terms of the Φ fields are non-local transformations. If we make the transformation

$$\Psi \rightarrow e^{i(\theta+\eta)} \Psi, \quad \bar{\Psi} \rightarrow e^{i(\theta-\eta)} \bar{\Psi} \quad (21)$$

then this is equivalent to making the transformation

$$\begin{pmatrix} \Phi \\ \Pi^* \end{pmatrix} \rightarrow e^{i\theta} \begin{pmatrix} \cos(\theta) & -\frac{1}{\omega(\nabla)} \sin(\eta) \\ \omega(\nabla) \sin(\eta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \Phi \\ \Pi^* \end{pmatrix}. \quad (22)$$

An alternative way of looking at this approximate $O(2) \otimes O(2)$ symmetry is to make a transformation to $\Psi', \bar{\Psi}'$ defined in the same way as before except that we replace ω by the mass m in (13) and (14). This version is mentioned by Haber and Weldon⁷ (see the sixth citation/footnote). Then these $\Psi', \bar{\Psi}'$ fields show only an approximate $O(2) \otimes O(2)$ symmetry, the symmetry being broken by terms of order \vec{k}/m as well as by the interactions in S_I but the symmetry is now a standard global symmetry in terms of the Φ and Π fields. Since the transformation from Φ to Ψ' is global, this approximate $O(2) \otimes O(2)$ symmetry is also a approximate global symmetry of the original Φ Lagrangian. It is not usually discussed because the terms which break the symmetry are large for relativistic particles.

So far everything has been exact, for any value of μ and T . In practice for a condensed matter system $m \simeq \mu \gg T$. The low temperature means that we can make the non-relativistic approximation

$$\tilde{S}_{0, \text{nr}}[\Psi^*, \Psi, \mu, \omega] \simeq \int_0^{-i\beta} dt \int d^3x \Psi^* \left[i \frac{\partial}{\partial t} + \frac{1}{2m} \nabla^2 + \mu - m \right] \Psi \quad (23)$$

The high particle density, or equivalently the large chemical potential, means that the mass gap term in the anti-particle $\bar{\Psi}$ free Hamiltonian dominates as $m - \mu \sim T \ll m \sim \mu$. This means that the quadratic part of the anti-particle Hamiltonian is to a good approximation $2m\bar{\Psi}^*\bar{\Psi}$. On integrating out the anti-particle fields we find that we just set $\bar{\Psi}^* = \bar{\Psi} = 0$ and so the anti-particle fields just drop out.

This then leaves us with

$$Z[j, j^*; \mu] \simeq Z_{\text{nr}} = \int \mathcal{D}\Psi^* \mathcal{D}\Psi \exp\left\{\int_0^{-i\beta} dt \int d^3x \Psi^* \left[i\frac{\partial}{\partial t} + \frac{1}{2m}\nabla^2 - m + \mu\right] \Psi + S_{I,\text{nr}}[\Psi^*, \Psi]\right\} \quad (24)$$

where for $\lambda|\Phi^*\Phi|^2$ we have $S_I[\Psi^*, \Psi] = \lambda(\Psi^*\Psi)^2/(4m^2)$. This means that we are now left with an approximate theory, (24), which also has an exact $O(2)$ symmetry. In terms of the original theory, this new $O(2)$ is broken by the terms of $O(k/m)$, $O((\mu - m)/m)$ which are dropped in reaching (24). It is a different symmetry from the original exact global $O(2)$ of the relativistic theory. The latter mixed particles and anti-particles, whereas (24) does not.

There is one caveat. The Φ -fields are periodic in imaginary time, but the Π -fields are not. This means that the Ψ -fields can only be approximately periodic, although when (24) is used exact periodicity is assumed.

The case of fermionic fields is rather different from this bosonic case as the Dirac equation is already first order in time. The problem with fermions is rather that one must generate the $\nabla^2/2m$ term from the first order $\vec{\gamma} \cdot \vec{\nabla}$ term, and also go from four by four Dirac gamma algebra to the two by two Pauli matrices. Essentially in this case one must follow a Foldy-Wouthuysen approach, but this will be considered elsewhere.

3. Chemical Potential

There are two approaches to the chemical potential in thermal field theory, which are distinguished by the way they split the physical information between initial conditions and microscopic dynamics - see table 2.

	Microscopic Dynamics	Initial Conditions
Method I	$H_e = H - \mu Q$	$\Phi_e(t_0) = \Phi_e(t_0 - i\beta)$
Method II	H	$\Phi(t_0) = e^{\beta\mu}\Phi(t_0 - i\beta)$

Table 2. Two Approaches to Chemical Potentials.

In Method I, the H and μQ terms of the density matrix are merged into a single an effective Hamiltonian, H_e which then explicitly depends on the chemical potential μ .

$$\rho = e^{-\beta(H - \mu Q)} = e^{-\beta H_e} \quad (25)$$

One of the great advantages of this method is that this new effective theory looks just like a theory with no chemical potential. This makes the boundary conditions the usual simple (anti-)periodic ones for bosonic (fermionic) fields so that all the usual well established methods can be called upon. Not surprisingly this approach is very popular^{7,8,9}. While the initial picture is simpler, it is often the case that new complications due to the presence of a chemical potential arise further down the road. All fields, and hence all Green functions, in Method I have a subscript e to denote that H_e is being used.

In Method II, all aspects of the equilibrium density matrix are treated in the same way. Thus one encodes both the temperature and the chemical potential in the boundary conditions, and the Hamiltonian used is physical one with no information about the density

matrix included. This leads to a rather more complicated boundary condition, which for bosonic fields is of the form

$$\Phi(t_0) = e^{\beta\mu}\Phi(t_0 - i\beta). \quad (26)$$

This makes matters more difficult than the zero chemical potential case from the very start. The advantage of this approach is that the density matrix is often best thought of as defining the initial conditions. The microscopic dynamics of the physical fields, as described by H , has nothing to do with the initial conditions. Thus keeping the Hamiltonian used free of contamination from the initial conditions means that Method II is a very physical approach. This is the approach used in the review of Landsman and van Weert¹¹. It is also interesting to note that it effectively the way that standard condensed matter texts^{12,13} deal with chemical potentials associated with fermionic degrees of freedom.

In fact for most out-of-equilibrium problems, the reverse is also true, that is the initial conditions need have nothing to do with the microscopic dynamics. Indeed in Closed Time Path approaches to non-equilibrium problems^{11,14} the initial conditions may always be encoded in terms of some ‘temperature’ and some ‘Hamiltonian’ as we can always write the density matrix as $\rho = \exp\{-\beta_{init}H_{init}\}$. However there is no reason why this H_{init} has anything to do the Hamiltonian which controls the microscopic interactions and hence evolution of the system.

Of course in many practical situations the initial conditions are related to some physical Hamiltonian. Equilibrium is an extreme case where the same H appears in initial density matrix and evolution. However, Method I is still using a mathematical trick to ‘simplify’ matters but it is doing it at the expense of physical clarity. The discussion above suggests that problems with Method I may occur when looking at dynamical problems. This is because H not H_e really controls the microscopic evolution yet in Method I H_e is used for everything. We will now justify this assertion in detail by looking first at the propagators which appear in both methods, i.e. we will solve simplest problem, that of a single free complex scalar field.

We will start by following Method I^{8,9,10}, which is to use real fields ϕ_{e1}, ϕ_{e2} , where $\Phi_e = (\phi_{e1} + i\phi_{e2})/\sqrt{2}$. Thus the action, S_e , is of the form

$$\begin{aligned} S_e = \int dt \int d^3\vec{x} & \left[\frac{1}{2} \left(\frac{\partial\phi_{e1}}{\partial t} + \mu\phi_{e2} \right)^2 - \frac{1}{2}(\nabla\phi_{e1})^2 - \frac{1}{2}m^2\phi_{e1}^2 \right. \\ & \left. + \frac{1}{2} \left(\frac{\partial\phi_{e2}}{\partial t} - \mu\phi_{e1} \right)^2 - \frac{1}{2}(\nabla\phi_{e2})^2 - \frac{1}{2}m^2\phi_{e2}^2 + O(\lambda) \right] \end{aligned} \quad (27)$$

$$= \int dt dt' (\phi_{e1}(t), \phi_{e2}(t))_a D_{ab}(t; t') \begin{pmatrix} \phi_{e1}(t') \\ \phi_{e2}(t') \end{pmatrix}_b + O(\lambda). \quad (28)$$

The inverse matrix propagator, D^{-1} , is given in energy-momentum coordinates, less an overall conservation delta function, by

$$D^{-1}(k_0 = 2\pi i n/\beta, \vec{k}) = \begin{pmatrix} k_0^2 - \omega^2 + \mu^2 & 2\mu i k_0 \\ -2\mu i k_0 & k_0^2 - \omega^2 + \mu^2 \end{pmatrix} \quad (29)$$

where for notational simplicity we have used the imaginary time form for the propagator. This propagator is not diagonal in the one and two fields but it is easy to find the two eigenvalues of D^{-1} which are $(k_0 + \mu)^2 - \omega^2$ and $(k_0 - \mu)^2 - \omega^2$. This suggests that there are four poles in the propagator⁹ at $k_0 = \pm\omega \pm \mu$. However, we started with a single complex scalar field so we expect only two poles. What has happened?

The solution is to look more carefully at the diagonalised Lagrangian. What we find is that the eigenstates are $(1, i)$ and $(1, -i)$ and hence the diagonalisation process has merely moved us from real fields ϕ_{e1}, ϕ_{e2} back to the original complex fields Φ_e, Φ_e^* . This is not surprising as we know that the latter are eigenstates of the charge operator, the real fields are not. It makes more sense in this case (where the symmetry is unbroken) to stick with the Q eigenstates. If we write out the free action after we have diagonalised (28), we obtain

$$S_e = \frac{1}{2} \frac{1}{(2\pi)^4} \sum_n \int d^3\vec{k} (\Phi_e^*, \Phi_e)_a \begin{pmatrix} (k_0 + \mu)^2 - \omega^2 & 0 \\ 0 & (k_0 - \mu)^2 - \omega^2 \end{pmatrix}_{ab}^{-1} \begin{pmatrix} \Phi_e \\ \Phi_e^* \end{pmatrix}_b \quad (30)$$

$$= -\frac{1}{\beta V} \int_0^{-i\beta} dt dt' \left(\frac{1}{2} \Phi_e^*(t) G_e^{-1}(t, t') \Phi_e(t') + \frac{1}{2} \Phi_e(t) H_e^{-1}(t, t') \Phi_e^*(t') \right) \quad (31)$$

$$= -\frac{1}{\beta V} \int_0^{-i\beta} dt dt' \Phi_e^*(t) G_e^{-1}(t, t') \Phi_e(t') \quad (32)$$

where the two propagators G and H are defined and related through

$$iG_e(t, t') = \text{Tr}\{e^{\beta(H-\mu Q)} T[\Phi_e(t) \Phi_e^*(t')]\} \quad (33)$$

$$iH_e(t, t') = \text{Tr}\{e^{\beta(H-\mu Q)} T[\Phi_e^*(t) \Phi_e(t')]\} = iG_e(t', t) \quad (34)$$

$$G_e^{-1}(k_0, \vec{k}) = [(k_0 + \mu)^2 - \omega^2] = H_e^{-1}(-k_0, \vec{k}) \quad (35)$$

where all the times are time ordering are Euclidean valued, and the energies are taken at $k_0 = 2\pi n/\beta$ where n is an integer. Thus we see from (30) that what appears to be four poles is actually the same two poles repeated a second time but with the time or energy argument reversed, as (32) shows. In fact we can obtain (32) much quicker by working with Φ, Φ^* from the start. This should not be surprising as these are the fields which are the eigenstates of the charge operator. Anyway there are only two distinct poles as one expects given that we have a single complex scalar field. For G_e , (33), the poles are located at

$$k_0 = \pm\omega - \mu. \quad (36)$$

Now let us turn to Method II and compare the propagator found there with (33). Essentially all we have to do is solve the Klein-Gordon equation subject to the weighted periodic boundary condition (26). Of course one needs other boundary conditions but these are provided by the equal time commutation relations. It is simple to look up the answer¹¹ and the time-ordered two-point Green function, G_t , is given by

$$\begin{aligned} iG_t(t, x; t', x') &= \text{Tr}\{e^{\beta(H-\mu Q)} T[\Phi(t, x) \Phi^*(t', x')]\} \\ &= \frac{1}{(2\pi)^4} \int d^4k \exp\{-ik_0 t + i\vec{k} \cdot \vec{x}\} \left[\frac{i}{k_0^2 - \omega^2 + i\epsilon} + \right. \\ &\quad \left. \frac{1}{\exp\{\beta(k_0 - \mu)\} - 1} \frac{2\pi\delta(k_0 - \omega)}{2\omega} + \frac{1}{\exp\{\beta(-k_0 - \mu)\} - 1} \frac{2\pi\delta(k_0 + \omega)}{2\omega} \right] \end{aligned} \quad (37)$$

Thus we see clearly that in Method II there are only two poles and they are at the same place as they are for a free field at zero density and arbitrary temperature, namely at

$$k_0 = \pm\omega. \quad (38)$$

So in Method II we find that the poles of the free propagator do not depend on any of the many-body parameters, and are only dependent on the form of the quadratic part of the

Hamiltonian. Thus the position of the poles depends only on microscopic physics which is described by the true physical Hamiltonian. On the other hand, the weighting or residue of the poles does depend on the initial conditions. Here the real-time propagator shows us this physics very clearly. We have a zero temperature term which is a pure quantum fluctuation contribution so has no explicit μ or β . Then there is one particle and one anti-particle pole, each weighted by the appropriate statistical distribution. These represent statistical fluctuations due to the real particles in the heat bath. This is why the particle pole has a much larger residue than the anti-particle pole when there are many particles but few anti-particles (large positive μ). So Method II leaves us with a picture which fits our physical intuition.

However, we now have an apparent contradiction between the results of Method I (36) and Method II (38). As the investigation of the relationships between various Green functions and the Green functions calculated in different formalisms¹⁵ showed, it is imperative that we are extremely careful in identifying the Green functions we are studying.

It is immediately obvious that the difference encountered here between Methods I and II is not a result of looking at different types Green functions. In the analysis of Method I we looked at Euclidean two-point Green functions and, implicitly, their analytic continuations, the retarded and advanced propagators. It is well known though that the real parts of these two-point functions are identical to the real parts of the time-ordered Green function, which was considered when looking at Method II^{11,12,13}. So we must go back to basics to find the source of the contradiction.

As we are looking at a propagator in Method I, let us write down the generating functional

$$Z_e[j_e^*, j_e] = \text{Tr}\{e^{-\beta(H-\mu Q)} T[e^{-i \int (j_e^* \Phi_e + \Phi_e^* j_e)}]\} \quad (39)$$

$$= \sum_{\Phi(\vec{x})} H_e \langle \Phi, t_0 - i\beta | T e^{-i \int (j_e^* \Phi_e + \Phi_e^* j_e)} | \Phi, t_0 \rangle \quad (40)$$

where we have used the unphysical effective Hamiltonian $H_e = H - \mu Q$ of Method I to evolve the bra. This means that in order to calculate Z_e , say by using path integrals, we have to evolve all fields and states using H_e not the physical H . Thus all the fields which appear in Z_e , or its derivatives, satisfy in the Heisenberg picture

$$\Phi_e(t) = e^{iH_e t} \Phi(0) e^{-iH_e t}. \quad (41)$$

The fields appearing in Method I are therefore *not* the same for an arbitrary time as the physical fields Φ , which satisfy in the Heisenberg picture

$$\Phi(t) = e^{iHt} \Phi(0) e^{-iHt}. \quad (42)$$

This is the reason we have been carefully labeling the fields used in Method I with the e subscript. Likewise, the Green functions must be distinguished.

There is however a simple link between the two fields, and we see that from (41)

$$\Phi_e(t) = e^{i(H-\mu Q)t} \Phi(0) e^{-i(H_e-\mu Q)t} \quad (43)$$

$$= e^{-i\mu Q t} e^{iHt} \Phi(0) e^{-iHt} e^{i\mu Q t} \quad (44)$$

$$= e^{-i\mu Q t} \Phi(t) e^{i\mu Q t} \quad (45)$$

$$\Phi_e(t) = e^{iq\mu t} \Phi(t). \quad (46)$$

where we have assumed that our field has a charge of q . The same connection is also noted elsewhere¹² but there is no discussion of Method II as a practical calculational tool for bosons.

Now, to make a link between the Green functions, we need to make a connection between the generating functionals in Method I and II. So we choose the sources such that

$$j_e(t) = e^{-i\mu t} j(t). \quad (47)$$

as this then gives

$$Z[j^*, j] = Z_e[j_e^*, j_e]. \quad (48)$$

The generating functional in Method I, Z_e , was given in (50) and that of Method II is

$$Z[j^*, j] = \text{Tr} e^{-\beta(H-\mu Q)} T e^{-i \int (j^* \Phi + \Phi^* j)} \quad (49)$$

$$= \sum_{\Phi(\vec{x})} {}_H \langle \Phi, t_0 - i\beta | T e^{-i \int (j^* \Phi + \Phi^* j)} | \phi, t_0 \rangle \quad (50)$$

This means that Method II generates *physical* Green functions e.g.

$$\Gamma^{(2N)}(t_1, \dots, t_N; s_1, \dots, s_N) = \frac{\partial^{(2N)} Z}{\partial j^*(t_1) \dots \partial j^*(t_N) \partial j(s_1) \dots \partial j(s_N)} \quad (51)$$

$$= \text{Tr} \left[e^{-\beta(H-\mu Q)} T [\Phi(t_1) \dots \Phi(t_N) \Phi^*(s_1) \dots \Phi(s_N)] \right] \quad (52)$$

where the T indicates path ordering appropriate for whatever finite temperature formalism is being considered. Likewise we see that with Method I we generate thermal expectation values but of the *unphysical* fields Φ_e, Φ_e^* , namely

$$\Gamma_e^{(2N)}(t_1, \dots, t_N; s_1, \dots, s_N) = \frac{\partial^{(2N)} Z}{\partial j_e^*(t_1) \dots \partial j_e^*(t_N) \partial j_e(s_1) \dots \partial j_e(s_N)} \quad (53)$$

$$= \text{Tr} \left[e^{-\beta(H-\mu Q)} T [\Phi_e(t_1) \dots \Phi_e(t_N) \Phi_e^*(s_1) \dots \Phi_e(s_N)] \right] \quad (54)$$

Now we can use (46), (47) and (48) to show how the unphysical Green functions of Method I, Γ_e of (54), are linked to the physical ones of Method II, Γ of (52). We find

$$\Gamma^{(2N)}(t_1, \dots, t_N; s_1, \dots, s_N) = e^{-i\mu \sum_j (t_j - s_j)} \Gamma_e^{(2N)}(t_1, \dots, t_N; s_1, \dots, s_N) \quad (55)$$

In particular, we find that for the propagators we have that

$$D(t, s) = \text{Tr} \left[e^{-\beta(H-\mu Q)} T [\Phi(t) \Phi^*(s)] \right] \quad (56)$$

$$= e^{-i\mu(t-s)} D_e(t, s) \quad (57)$$

$$D(k_0) = D_e(k_0 - \mu) \quad (58)$$

These relations apply to any approximation to the full Green functions and in particular for the free field case. This resolves the paradox we found above, (36) vs. (38). Equation (58) tells us that the poles in Method I and Method II are related by a simple shift of μ in the energy.

It is important to note from the relation between the two generating functions that the partition function is identical in both cases

$$Z = Z[0, 0] = Z_e[0, 0] \quad (59)$$

Thus both methods give the same answer for the partition function and so will give answers directly for all thermodynamic quantities. Indeed, any static quantity, i.e. time independent

or equivalently zero energy quantity, can be extracted directly from either method. Only when one has time dependence or is looking at non-zero energy does one need to use formulae such as (55) to move between the two formalisms.

Finally we mention a complication with symmetry breaking, or equivalently Bose-Einstein condensation. This area has long been of interest^{4,7,8,9,13} and has had a lot of recent attention as a result of the experimental observation of Bose-Einstein condensation in “Quantum Atom” systems¹⁶. So far we have been working in a situation where the symmetry is unbroken, $\langle \Phi \rangle = 0$. However what happens if we are in a phase where this is not true? In Method I with periodic boundary conditions we proceed as usual and shift the fields

$$\Phi_e(t, \vec{x}) = v_e + \Psi_e(t, \vec{x}) \quad (60)$$

$$\langle \Phi_e \rangle = v_e \quad \langle \Psi_e \rangle = 0, \quad (61)$$

where v_e is a constant in the usual way. However in Method II such a simple approach is not compatible with the more complicated boundary conditions (26). By using the link between the fields in Methods I and II, (46) we see that we have to use

$$\Phi(t, \vec{x}) = v(t) + \Psi(t, \vec{x}) \quad (62)$$

$$\langle \Phi_e \rangle = v(t), \quad v(t) = e^{q\mu t} v_e, \quad \langle \Psi_e \rangle = 0. \quad (63)$$

4. Conclusions

We have seen how to link the fields used in relativistic and non-relativistic theories. Equation (46) gives this link. Note that we worked within Method I but the formula works for Method II as well.

I have also discussed how to link the two approaches used when including chemical potentials. In Method I, fields Φ_e are used which evolve according to $H_e = H - \mu Q$. This means that for these fields, we are measuring the energy on a scale shifted by μ relative to the standard relativistic field theory definition. Thus Method I is actually quite close to Method II. Method I is only ‘unphysical’ in the sense that it is measuring its energies from a zero which is shifted from the standard of relativistic field theory. Method I is favoured by condensed matter physicists when dealing with bosonic systems¹². It is also used in relativistic texts for calculations of partition functions^{8,9,10}. It is easy to use for free energies because of the equality (59) between the partition function of Method I, and that of method II.

However, for dynamical quantities there is a difference between Methods I and II, summarised by (55). In this case, Method II is clearer in my opinion as it sticks rigidly to the conventions of relativistic field theory. In particular all energies are measured relative to the same zero, which is the energy of the vacuum state (in a simple unbroken case). However the same physical information can be obtained from Method II. The problem is that energies are then measured relative to a zero shifted by $q\mu$ from that of Method I and the standard relativistic zero. This means that fields with different charges under the various symmetry groups involved will have their zero’s of energy shifted by different amounts! Thus in trying to understand the electroweak model one runs a serious risk of getting confused about the energies involved of various modes. So in discussions of dynamics, poles of propagators or whatever, in a relativistic field particle context but where Method I is used, such as in Bernstein and Dodelson⁹, one must be very careful that one understands that the energy zero has been shifted. Shifting your zero of energy does not mean that anything interesting has happened physically!

However it is a matter of personal taste which method one chooses to use. The same physical information is contained in both approaches. One must just be very careful when interpreting the information and be very clear what energy zero one is working with. Method I is closest to the way condensed matter physicists work with bosonic systems, with energies measured relative to the Fermi surface (which is equal to μ at zero temperature). Method II means that we stick to the conventions used amongst relativistic particle physicists.

5. Acknowledgements

I would like to thank the Royal Society for their support through a University Research Fellowship. I have benefited greatly from discussions with G. Karra, A.J.Gill, R.J.Rivers and D.A.Steer, with whom various aspects of the work discussed here, or extensions thereof, have been developed at Imperial College. I thank Dr. F.C.Khanna for enlightening discussions and hospitality at the Theoretical Physics Institute of the University of Alberta, Edmonton, Canada. I would finally like to thank the organisers for such an enjoyable meeting.

1. W.H. Zurek, *Nature* **317**, (1985)505, *Acta Physica Polonica* **B24**, (1993) 1301; *Cosmological Experiments in Condensed Matter Systems*, Los Alamos preprint LA-UR 95-2269 (submitted to Physics reports).
2. I.Chuang, R.Durrer, N.Turok and B.Yurke, *Science* **251** (1991) 1336.
3. R.J. Rivers *Vortex Production at Phase Transitions in Non-Relativistic and Relativistic Media*, to be published in the proceedings of the 3me. *Colloque Cosmologie, Paris, June 1995* (World Scientific Press) 1996; see also these proceedings.
4. H.Chu, Y.C.Chang, H.Umezawa and F.C.Khanna, *Phys. Lett.***A201** (1995) 353.
5. P.C.Hendry, N.S.Lawson, R.A.M.Lee, P.V.E.McClintock and C.D.H.Williams, *Nature* **368**, (1994) 315; V.M.H.Ruutu, M.Krusius, U.Parts and G.E.Volovik, *Neutron Mediated Vortex Nucleation in Rotating Superfluid $^3\text{He} - B$* , Low Temperature Laboratory, Helsinki University of Technology (in preparation).
6. A.J.Gill and R.J.Rivers, *Phys. Rev.***D51** (1995) 6949.
7. H.E.Haber and H.A.Weldon, *Phys. Rev.***D25** (1982) 502.
8. J.I. Kapusta, *Finite Temperature Field Theory* (Cambridge University Press, Cambridge, 1989).
9. J.Bernstein, S.Dodelson, *Phys. Rev. Lett.***66** (1991) 683.
10. K.M.Benson, J.Bernstein, S.Dodelson, *Phys. Rev.***D44** (1991) 2480.
11. N.P. Landsman and Ch.G. van Weert, *Phys. Rep.* **145** (1987) 141.
12. A.A. Abrikosov, L.P. Gor'kov and I.Ye. Dzyaloshinskii, *Quantum Field Theoretical Methods in Statistical Physics* (Pergamon Press, Oxford, 1965).
13. A. Fetter and J. Walecka, *Quantum Theory of Many-Particle Systems* (McGraw-Hill, New York, 1971);
14. K-C. Chou, Z-B. Su, B-L. Hao and L. Yu, *Phys. Rep.* **118** (1985) 1.
15. T.S.Evans, *Nucl. Phys.***B374** (1992) 340; in *Thermal Field Theories*, ed. H. Ezawa, T. Arimitsu and Y. Hashimoto (North Holland, Amsterdam, 1991), p.441 .
16. M.H.Anderson, J.R.Ensher, M.R.Mathews, C.E.Wieman and E.A.Cornell, *Science* **269** (1995) 198.